Mathematical Tools for Physics

Quadratic Equation

An equation of second degree is called a quadratic equation. It is of the form :- $ax^2 + bx + c = 0$

The roots of a quadratic equation are $X = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Binomial Theorem

If n is any integer, positive or negative or a fraction and x is any real number, then

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$

If |x| <<1 , then $(1+x)^n = 1 + nx$.

Mensuration

1. Area of a circle = $\pi r^2 = \pi D^2/4$
2. Surface area of a sphere = $4\pi r^2 = \pi D^2$
3. Volume of a sphere = $(4/3)\pi r^3$
4. Surface area of a cylinder = $2\pi r(r+l)$
5. Volume of a cylinder = $\pi r^2 l$
6. Curved surface area of a cone = πrl
7. Volume of a cone = $(1/3)\pi r^2 h$
8. Surface area of a cube = $6 \times (\text{side})^2$
9. Volume of a cube = $(\text{side})^3$

Fundamental Trigonometric relations

$\text{Cosec}\theta = \frac{1}{\sin\theta}$ $\text{Sec}\theta = \frac{1}{\cos\theta}$ $\text{Cot}\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$ $\text{Tan}\theta = \frac{\sin\theta}{\cos\theta}$

$\text{Sin}^2\theta + \text{Cos}^2\theta = 1$ $1 + \tan^2\theta = \text{Sec}^2\theta$ $1 + \text{Cot}^2\theta = \text{Cosec}^2\theta$

Sin(A+B) = SinACosB + CosASinB Sin (A – B) = SinACosB – CosASinB

Cos (A + B) = CosACosB – SinASinB Cos (A – B) = CosACosB + SinASinB

$\text{Tan}(A\pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Sin 2A = 2SinACosA $\text{Cos}\,2A = 2\text{Cos}^2A - 1 = 1 - 2\text{Sin}^2A = \text{Cos}^2A - \text{Sin}^2A$

Sin (A+B) + Sin(A-B) = 2 SinACosB $\text{Sin}\,C + \text{Sin}\,D = 2\,\text{Sin}\frac{C+D}{2}\,\text{Cos}\frac{C-D}{2}$

Cos(A+B) + Cos (A-B) = 2CosACosB $\text{Cos}\,C + \text{Cos}\,D = 2\text{Cos}\frac{C+D}{2}\,\text{Cos}\frac{C-D}{2}$

x	$-\theta$	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\Pi-\theta$	$\Pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\Pi-\theta$	$2\Pi+\theta$
sinx	-sinθ	cosθ	cosθ	sinθ	-sinθ	-cosθ	-cosθ	-sinθ	sinθ
cosx	cosθ	sinθ	-sinθ	-cosθ	-cosθ	-sinθ	sinθ	cosθ	cosθ
tanx	- tanθ	cotθ	-cotθ	-tanθ	tanθ	cotθ	-cotθ	-tanθ	tanθ

Logarithms

$\text{Log}_a mn = \text{Log}_a m + \text{Log}_a n$ $\text{Log}_a\left(\frac{m}{n}\right) = \text{Log}_a m - \text{Log}_a n$ $\text{Log}_a m^n = n\text{Log}_a m$

$\text{Log}_a m = \text{Log}_b m \times \text{Log}_a b$ $\text{Log}_a a^n = n$ $\text{Log}_a 1 = 0$

$\text{Log}_a a = 1$ $a^x = e^{x \log_e a}$

Approximate Values

If angle (θ) is very small then i.e. θ→0, then Sin θ ≅ θ; Cos θ ≅ 1 and Tan θ ≅ θ

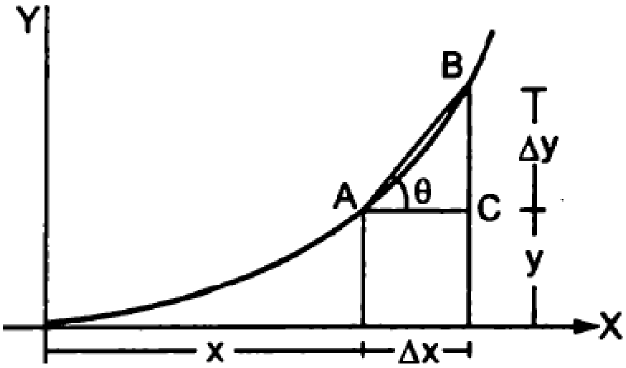
Differential Calculus

→ <u>Derivatives</u>: The derivative of f at x is denoted by f′(x) or $\frac{d}{dx}$ f(x).

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.}$$

Derivative of f(x) at a is denoted by f ′(a).

→ $\frac{dy}{dx}$ <u>AS RATE MEASURER</u>: Consider two quantities y and x interrelated in such a way that for each value of x there is one and only one value of y. (Figure represents the graph of y versus x.)



When x changes by Δx, y changes by Δy so that the rate of change seems to be equal to $\frac{\Delta y}{\Delta x}$.
If A be the point (x, y) and B be the point (x + Δx, y + Δy), the rate $\frac{\Delta y}{\Delta x}$ equals the slope of the line AB. We have

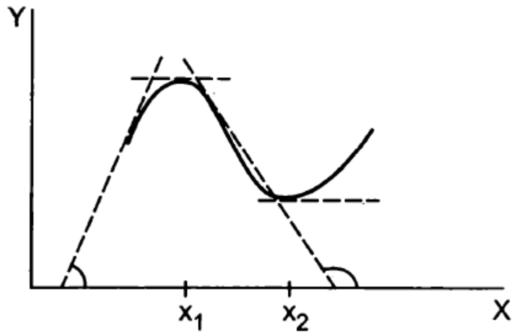
$$\frac{\Delta y}{\Delta x} = \frac{BC}{AC} = \tan\theta.$$

However, this cannot be the precise definition of the rate. Because the rate also varies between the points A and B. The curve is steeper at B than at A. Thus, to know the rate of change of y at a particular value of x, say at A, we have to take very small Δx. However small we take Δx, as long as it is not zero the rate may vary within that small part of the curve. However, if we go on drawing the point B closer to A and every time calculate $\frac{\Delta y}{\Delta x}$ = tanθ, we shall see that as Δx is made smaller and smaller, the slope tanθ of the line AB approaches the slope of the tangent at A. This slope of the tangent at A thus gives the rate of change of y with respect to x at A. This rate is denoted by $\frac{dy}{dx}$ .Thus,

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}}$$

Note that if the function y increases with an increase in x at a point, $\frac{dy}{dx}$ is <u>positive</u> there, because both Δy and Δx are positive. If the function y decreases with an increase in x, Δy is negative when Δx is positive. Then and hence $\frac{dy}{dx}$ is <u>negative</u>.

→ <u>MAXIMA AND MINIMA:</u> Suppose a quantity y depends on another quantity x in a manner shown in figure. It becomes maximum at x_1 and minimum at x_2 .



At these points the tangent to the curve is parallel to the X-axis and hence its slope is tan θ = 0. But the slope of the curve y-x equals the rate of change $\frac{dy}{dx}$
Thus, at a maximum or a minimum, $\frac{dy}{dx} = 0$.
Just before the maximum the slope is positive, at the maximum it is zero and just after the maximum it is negative. Thus, $\frac{dy}{dx}$ decreases at a maximum and hence the rate of change of $\frac{dy}{dx}$ is negative at a maximum i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} < 0$ at a maximum.
The quantity $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the rate of change of the slope. It is written as $\frac{d^2y}{dx^2}$ Thus,

$$\boxed{\text{The condition of a maximum is } \frac{dy}{dx} = 0 \ \& \ \frac{d^2y}{dx^2} < 0}$$

Similarly, at a minimum the slope changes from negative to positive. The slope increases at such a point and hence $\frac{d^2y}{dx^2} > 0$.

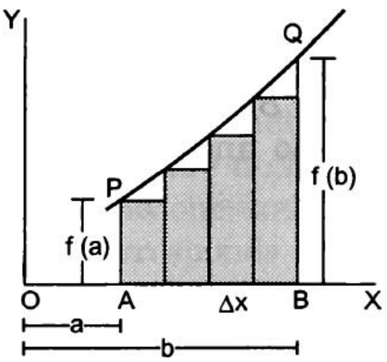
$$\boxed{\text{The condition of a minimum is } \frac{dy}{dx} = 0 \ \& \ \frac{d^2y}{dx^2} > 0}$$

→ Differential Formulae:

1. If C is a constant then $\frac{dc}{dx} = 0$; $\frac{dcy}{dx} = c\frac{dy}{dx}$
2. $\frac{d(x^n)}{dx} = nx^{n-1}$
3. $\frac{de^x}{dx} = e^x$
4. $\frac{da^x}{dx} = a^x \log_e a$
5. $\frac{d\log_e x}{dx} = \frac{1}{x}$
6. $\frac{d\log_a x}{dx} = \frac{1}{x}\log_e a$
7. $\frac{d\sin x}{dx} = \cos x$
8. $\frac{d\cos x}{dx} = -\sin x$
9. $\frac{d\tan x}{dx} = \sec^2 x$
10. $\frac{d\cot x}{dx} = -\operatorname{cosec}^2 x$
11. $\frac{d\operatorname{cosec} x}{dx} = -\operatorname{cosec}x.\cot x$
12. $\frac{d\sec x}{dx} = \sec x.\tan x$
13. $\frac{df(x) \pm g(x)}{dx} = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$
14. $\frac{df(x)\,g(x)}{dx} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$
15. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{df(x)}{dx} - f(x)\frac{dg(x)}{dx}}{(g(x))^2}$
16. $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dz}.\frac{dz}{dx}$ (chain rule)

Integral Calculus

Let y = f(x) and PQ be a curve representing the relation between quantities x and y (fig). The point P corresponds to x = a and Q corresponds to x = b.
Let us divide the length AB in N equal elements each of length $\Delta x = \frac{b-a}{N}$. From the ends of each small length we draw lines parallel to the Y-axis. From the points where these lines cut the given curve, we draw short lines parallel to the X-axis. This constructs the rectangular bars shown shaded in the figure. The sum of the areas of these N rectangular bars differs slightly from the area PABQ. This difference is the sum of the small triangles formed just under the curve. Now the important point is the following. As we increase the number of intervals N, the vertices of the bars touch the curve PQ at more points and the total area of the small triangles decreases. As N tends to infinity (Δx tends to zero because $\Delta x = \frac{b-a}{N}$) the vertices of the bars touch the curve at infinite number of points and the total area of the triangles tends to zero. In such a limit the sum becomes the area of PABQ. Thus, we may write,



The area PABQ = f(a) Δx + f(a + Δx) Δx + f(a + 2Δx) Δx + ... + f [a + (N - 1) Δx] Δx

$= \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)\Delta x$; where X_i takes the values a, a + Δx, a + 2Δx,... b - Δx.

$= \int_a^b f(x)dx = [F(b) - F(a)]$

In mathematics $\int_a^b f(x)dx$ is read as the integral of f(x) with respect to x within the limits x = a to x = b. Here a is called the lower limit and b the upper limit of integration. The integral is the sum of a large number of terms of the type f(x) Δx with x continuously varying from a to b and the number of terms tending to infinity.
In mathematics, special methods have been developed to find the integration of various functions f (x). If we can find such a function F(x) such that the derivative of F(x) is f(x) that is, $\frac{dF(x)}{dx} = f(x)$,
then $\int f(x)dx = F(x) + c$ and $\int_a^b f(x)dx = [F(b) - F(a)]$

Integral Formulae:

1. $\int dx = x + c$ Where c = constant
2. $\int x^n dx = \frac{x^{n+1}}{n+1} + c$
3. $\int \frac{dx}{x} = \log_e x + c$
4. $\int \sin x\, dx = -\cos x + c$
5. $\int \sin ax\, dx = \frac{-\cos ax}{a} + c$
6. $\int \cos x\, dx = \sin x + c$
7. $\int \sec^2 x\, dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
9. $\int \sec x \tan x\, dx = \sec x + c$
10. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
11. $\int e^x dx = e^x + c$
12. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
13. If $\int f(x)dx = F(x) + c$ then, $\int f(ax+k)\, dx = \frac{F(ax+k)}{a} + c$; a, b,c and k are constants
14. If $\int f(x)dx = F(x) + c$ then, $\int_a^b f(x)dx = [F(b) - F(a)]$; a, b,c and k are constants
15. $\int (u \pm v \pm w \pm \ldots)dx = \int u dx \pm \int v dx \pm \int w dx \pm \ldots$..
16. $\int u.v\, dx = u\int v dx + \int (\int v dx)\left(\frac{du}{dx}\right) dx$

Scalar and vector quantities:

A study of motion will involve the introduction of a variety of quantities, which are used to describe the physical world. Examples of such quantities are distance, displacement, speed, velocity, acceleration, mass, momentum, energy, work, power etc. All these quantities can be divided into two categories – scalars and vectors.
The scalar quantities have magnitude only. It is denoted by a number and unit. Examples: length, mass, time, speed, work, energy, temperature etc. Scalars of the same kind can be added, subtracted, multiplied or divided by ordinary laws.
The vector quantities have both magnitude and direction. Examples: displacement, velocity, acceleration, force, weight, momentum, etc.

- **Representation of a vector**

Vector quantities are often represented by a scaled vector diagrams. Vector diagrams represent a vector by the use of an arrow drawn to scale in a specific direction. An example of a scaled vector diagram is shown in Fig.
From the figure, it is clear that
(i) The scale is listed.
(ii) A line with an arrow is drawn in a specified direction.
(iii) The magnitude and direction of the vector are clearly labelled.
In the above case, the diagram shows that the magnitude is 4 N and direction is 30° to x-axis. The length of the line gives the magnitude and arrow head gives the direction. In notation, the vector is denoted in bold face letter such as **A** or with an arrow above the letter as $\vec{A}$, read as vector A or A vector while its magnitude is denoted by A or by $|\vec{A}|$.

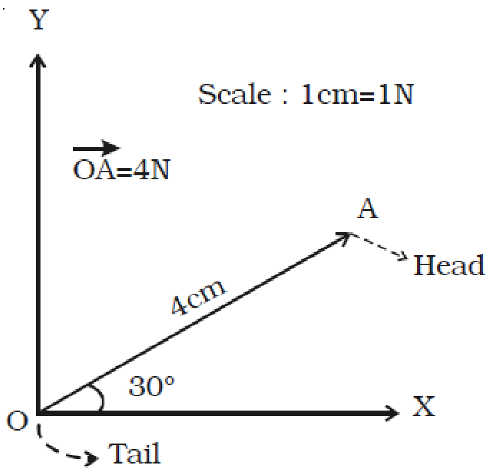
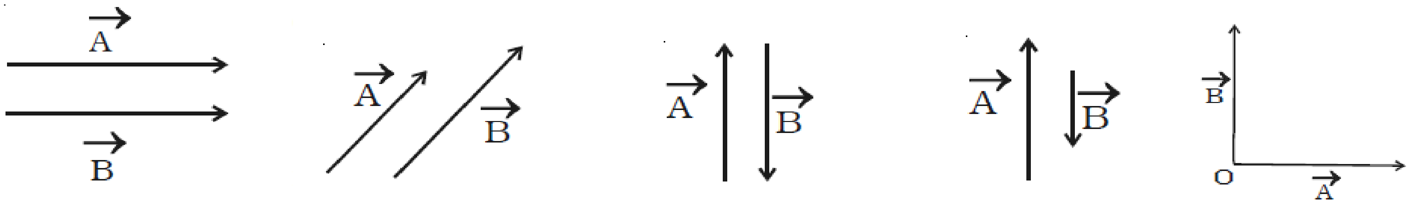


Fig Representation of a vector

- **Different types of vectors**



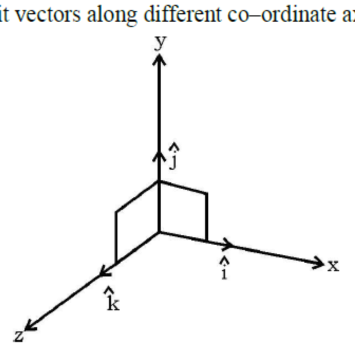
Equal vectors *Like vectors* *Opposite vectors* *Unlike Vectors* *Co-initial vectors*

(i) **Equal vectors**: Two vectors are said to be equal if they have the same magnitude and same direction, wherever be their initial positions. In Fig the vectors $\vec{A}$ and $\vec{B}$ have the same magnitude and direction. Therefore $\vec{A}$ and $\vec{B}$ are equal vectors.
(ii) **Like vectors**: Two vectors are said to be like vectors, if they have same direction but different magnitudes as shown in Fig.
(iii) **Opposite vectors**: The vectors of same magnitude but opposite in direction, are called opposite vectors (Fig.).
(iv) **Unlike vectors**: The vectors of different magnitude acting in opposite directions are called unlike vectors. In Fig. the vectors $\vec{A}$ and $\vec{B}$ are unlike vectors.
(v) **Unit vector:** A vector having unit magnitude is called a unit vector. It is also defined as a vector divided by its own magnitude. A unit vector in the direction of a vector $\vec{A}$ is written as $\hat{A}$ and is read as 'A cap' or 'A caret' or 'A hat'. Therefore, $\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|}$ (or) $\vec{\mathbf{A}} = |\vec{\mathbf{A}}|\ \hat{\mathbf{A}}$

Thus, a vector can be written as the product of its magnitude and unit vector along its direction.
A unit vector is unitless and dimensionless vector and represents direction only.

Orthogonal unit vectors

There are three most common unit vectors in the positive directions of X,Y and Z axes of Cartesian coordinate system, denoted by $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ respectively. Since they are along the mutually perpendicular directions, they are called orthogonal unit vectors.

unit vectors along different co–ordinate axis


(vi) **Null vector or zero vector**: A vector whose magnitude is zero, is called a null vector or zero vector. It is represented by $\vec{0}$ and its starting and end points are the same. The direction of null vector is not defined. Velocity of a stationary object, acceleration of an object moving with uniform velocity and resultant of two equal and opposite vectors are the examples of null vector.

(vii) **Proper vector**: All the non-zero vectors are called proper vectors.

(viii) **Co-initial vectors**: Vectors having the same starting point are called co-initial vectors. In Fig. $\vec{A}$ and $\vec{B}$ start from the same origin O. Hence, they are called as co-initial vectors.

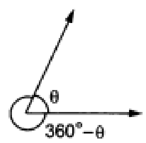
(ix) **Collinear Vectors:** Vectors having equal or unequal magnitudes but acting along the same or parallel lines are called collinear vectors.

Collinear Vector

(x) **Coplanar vectors**: Vectors lying in the same plane are called coplanar vectors and the plane in which the vectors lie are called plane of vectors.

- **Angle between vectors**

When two vectors are drawn with both the tails coinciding, two angles are formed between them (figure). One of the angles is smaller than 180° and the other is greater than 180° unless both are equal to 180°. The smaller angle considered as angle between vector (θ).



- **Addition of vectors**

As vectors have both magnitude and direction they cannot be added by the method of ordinary algebra. Vectors can be added graphically or geometrically. We shall now discuss the addition of two vectors graphically using head to tail method.

Consider two vectors $\vec{P}$ and $\vec{Q}$ which are acting along the same line. To add these two vectors, join the tail of $\vec{Q}$ with the head of $\vec{P}$ (Fig).

The resultant of is $\vec{R} = \vec{P} + \vec{Q}$. The length of the line AD gives the magnitude of $\vec{R}$. $\vec{R}$ acts in the same direction as that of $\vec{P}$ and $\vec{Q}$.

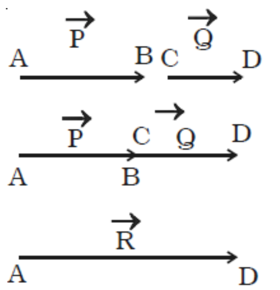
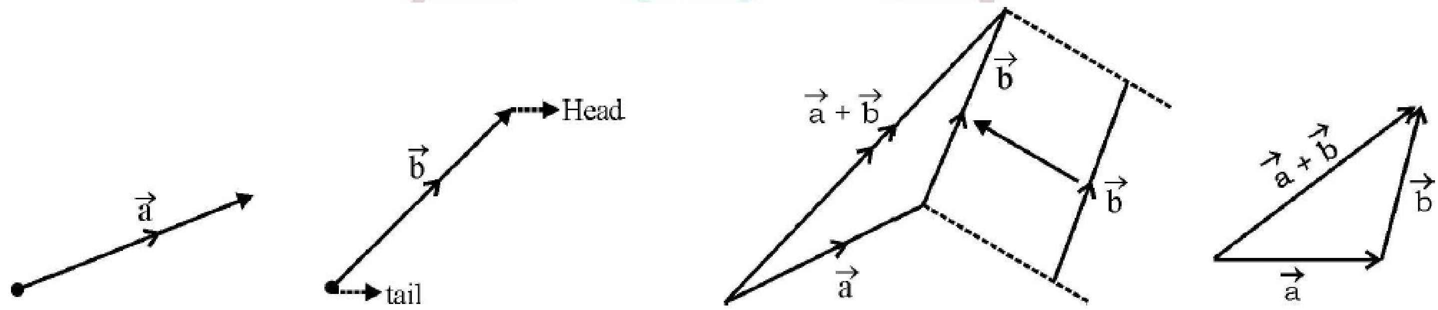


Fig. Addition of vectors

In order to find the sum of two vectors, which are inclined to each other, triangle law of vectors or parallelogram law of vectors, can be used.

(i) **Triangle law of vectors**



If two vectors are represented in magnitude and direction by the two adjacent sides of a triangle taken in order, then their resultant is the closing side of the triangle taken in the reverse order.

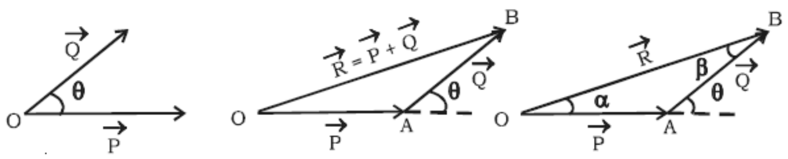


Fig. Triangle law of vectors

To find the resultant of two vectors $\vec{P}$ and $\vec{Q}$ which are acting at an angle θ , the following procedure is adopted.

First draw $\overrightarrow{OA} = \vec{P}$ (Fig.) Then starting from the arrow head of $\vec{P}$, draw the vector $\overrightarrow{AB} = \vec{Q}$. Finally, draw a vector $\overrightarrow{OB} = \vec{R}$ from the tail of vector $\vec{P}$ to the head of vector $\vec{Q}$. Vector $\overrightarrow{OB} = \vec{R}$ is the sum of the vectors $\vec{P}$ and $\vec{Q}$. Thus $\vec{R} = \vec{P} + \vec{Q}$. The magnitude of $\vec{P} + \vec{Q}$ is determined by measuring the length of $\vec{R}$ and direction by measuring the angle between $\vec{P}$ and $\vec{R}$.

The magnitude and direction of $\vec{R}$, can be obtained by using the sine law and cosine law of triangles. Let α be the angle made by the resultant $\vec{R}$ with $\vec{P}$. The magnitude of $\vec{R}$ is,

$R^2 = P^2 + Q^2 - 2PQ\cos(180^o - \theta)$

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

The direction of R can be obtained by, $\frac{P}{\sin\beta} = \frac{Q}{\sin\alpha} = \frac{R}{\sin(180° - \theta)} = \frac{R}{\sin\theta}$

(ii) **Parallelogram law of vectors**

If two vectors acting at a point are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal passing through the common tail of the two vectors.

Let us consider two vectors $\vec{P}$ and $\vec{Q}$ which are inclined to each other at an angle θ as shown in Fig.. Let the vectors $\vec{P}$ and $\vec{Q}$ be represented in magnitude and direction by the two sides OA and OB of a parallelogram OACB. The diagonal OC passing through the common tail O, gives the magnitude and direction of the resultant $\vec{R}$.

CD is drawn perpendicular to the extended OA, from C. Let ∠COD made by $\vec{R}$ with $\vec{P}$ be α.

From right angled triangle OCD,

$OC^2 = OD^2 + CD^2 = (OA + AD)^2 + CD^2 = OA^2 + AD^2 + 2.OA.AD + CD^2$

$OC^2 = OA^2 + AD^2 + CD^2 + 2.OA.AD$...(1)

In Fig. ∠BOA = θ = ∠CAD

From right angled Δ CAD, $AC^2 = AD^2 + CD^2$ ...(2)

Substituting (2) in (1) $OC^2 = OA^2 + AC^2 + 2OA.AD$ ...(3)

From Δ ACD,

CD = AC sin θ ...(4)

AD = AC cos θ ...(5)

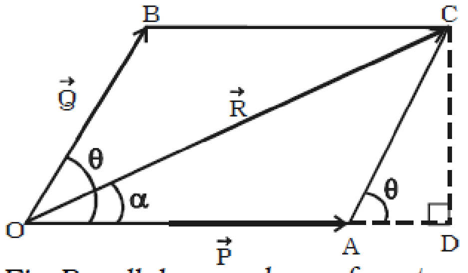


Fig Parallelogram law of vectors

Substituting (5) in (3) $OC^2 = OA^2 + AC^2 + 2\ OA.AC\cos\theta$

Substituting OC = R, OA = P,

OB = AC = Q in the above equation

$R^2 = P^2 + Q^2 + 2PQ\cos\theta$ (or) $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$ ...(6)

Equation (6) gives the magnitude of the resultant. From Δ OCD,

$\tan\alpha = \frac{CD}{OD} = \frac{CD}{OA+AD}$

Substituting (4) and (5) in the above equation,

$\tan\alpha = \frac{AC\sin\theta}{OA+AC\cos\theta} = \frac{Q\sin\theta}{P+Q\cos\theta}$ (or) $\alpha = \tan^{-1}\left(\frac{Q\sin\theta}{P+Q\cos\theta}\right)$ ...(7)

Equation (7) gives the direction of the resultant.

<u>Special Cases:</u>

(i) When two vectors act in the same direction: In this case, the angle between the two vectors $\theta = 0^o$, $\cos 0^o = 1$, $\sin 0^o = 0$

From (6) $R = \sqrt{P^2 + Q^2 + 2PQ\cos 0^\circ} = \sqrt{P^2 + Q^2 + 2PQ} = (P + Q)$

From (7) $\alpha = \tan^{-1}\left(\frac{Q\sin 0^\circ}{P+Q\cos 0^\circ}\right) = \tan^{-1}(0)$ (i.e) $\alpha = 0$

Thus, the resultant vector acts in the same direction as the individual vectors and is equal to the sum of the magnitude of the two vectors.

(ii) When two vectors act in the opposite direction: In this case, the angle between the two vectors $\theta = 180°$, $\cos 180° = -1$, $\sin 180^o = 0$.

From (6) $R = \sqrt{P^2 + Q^2 - 2PQ} = (P - Q)$

From (7) $\alpha = \tan^{-1}(0)$ (i.e) $\alpha = 0$

Thus, the resultant vector has a magnitude equal to the difference in magnitude of the two vectors and acts in the direction of the bigger of the two vectors.

(iii) When two vectors are at right angles to each other: In this case, $\theta = 90°$, $\cos 90° = 0$, $\sin 90° = 1$

From (6) $R = \sqrt{P^2 + Q^2}$

From (7) $\alpha = \tan^{-1}\left(\frac{Q}{P}\right)$

The resultant $\vec{R}$ vector acts at an angle α with vector $\vec{P}$.

- **Subtraction of vectors**

The subtraction of a vector from another is equivalent to the addition of one vector to the negative of the other.

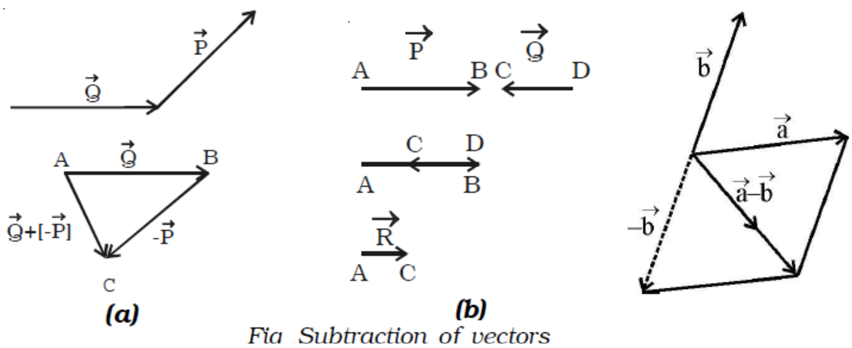
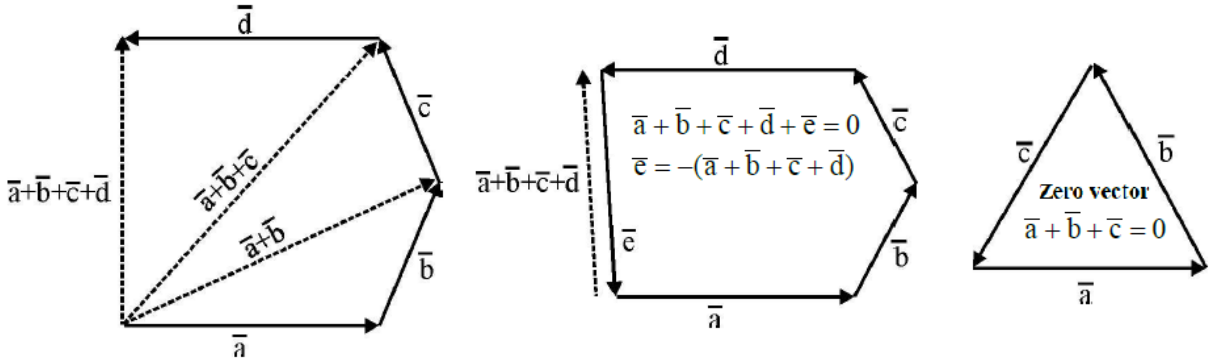


Fig Subtraction of vectors

For example $\vec{Q} - \vec{P} = \vec{Q} + (-\vec{P})$

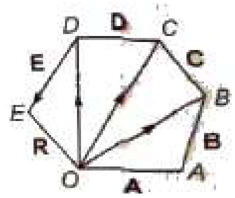
Thus to subtract $\vec{P}$ from $\vec{Q}$, one has to add $-\vec{P}$ with $\vec{Q}$ (Fig a). Therefore, to subtract $\vec{P}$ from $\vec{Q}$, reversed $\vec{P}$ is added to the $\vec{Q}$. For this, first draw $\overrightarrow{AB} = \vec{Q}$ and then starting from the arrow head of $\vec{Q}$, draw $\overrightarrow{BC} = (-\vec{P})$ and finally join the head of $-\vec{P}$. Vector $\vec{R}$ is the sum of $\vec{Q}$ and $-\vec{P}$. (i.e) difference $\vec{Q} - \vec{P}$.

The resultant of two vectors which are antiparallel to each other is obtained by subtracting the smaller vector from the bigger vector as shown in Fig b. The direction of the resultant vector is in the direction of the bigger vector.



#Polygon Law of Vectors:

It states that if number of vectors acting on a particle at a time are represented in magnitude and – direction by the various sides of an open polygon taken in same order, their resultant vector $\vec{R}$ is represented in magnitude and direction by the closing side of polygon taken in opposite order. In fact, polygon law of vectors is the outcome of triangle law of vectors.



$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$ or, $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$

#Properties of Vector Addition:

(i) Vector addition is commutative, i.e., $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

(ii) Vector addition is associative, i.e., $\vec{A} + (\vec{B} + \vec{C}) = \vec{B} + (\vec{C} + \vec{A}) = \vec{C} + (\vec{A} + \vec{B})$

(iii) Vector addition is distributive, i.e., $m(\vec{A} + \vec{B}) = m\vec{A} + m\vec{B}$

#Rotation of a Vector:

(i) If a vector is rotated through an angle θ, which is not an integral multiple of 2 π, the vector changes.

(ii) If the frame of reference is rotated or translated, the given vector does not change. The components of the vector may, however, change.

- **Product of a vector and a scalar**

Multiplication of a scalar and a vector gives a vector quantity which acts along the direction of the vector.

Examples

(i) If $\vec{a}$ is the acceleration produced by a particle of mass m under the influence of the force, then $\vec{F} = m\vec{a}$

ii) momentum = mass × velocity (i.e) $\vec{p} = m\vec{v}$.

- **Resolution of vectors and rectangular components**

A vector directed at an angle with the co-ordinate axis, can be resolved into its components along the axes. This process of splitting a vector into its components is known as resolution of a vector.

Consider a vector $\vec{R} = \overrightarrow{OA}$ making an angle θ with X - axis. The vector R can be resolved into two components along X - axis and Y-axis respectively. Draw two perpendiculars from A to X and Y axes respectively. The intercepts on these axes are called the scalar components R_x and R_y.

Then, OP is Rx, which is the magnitude of x component of $\vec{R}$ and OQ is R_y, which is the magnitude of y component of $\vec{R}$.

From Δ OPA,

$\cos\theta = \frac{OP}{OA} = \frac{R_x}{R}$ (or) $R_x = R\cos\theta$

$\sin\theta = \frac{OQ}{OA} = \frac{R_y}{R}$ (or) $R_y = R\sin\theta$

and $R^2 = R_x^2 + R_y^2$

Also, $\vec{R}$ can be expressed as $\vec{R} = R_x\vec{i} + R_y\vec{j}$ where $\vec{i}$ and $\vec{j}$ are unit vectors.

In terms of R_x and R_y, θ can be expressed as $\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$

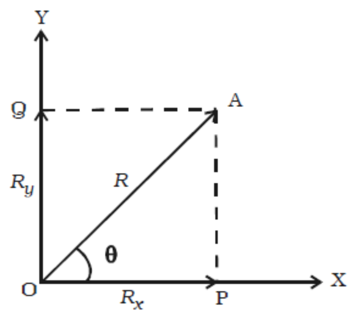


Fig. Rectangular components of a vector

A vector $\vec{A}$ can be resolved into component along two given vectors $\vec{a}$ and $\vec{b}$ lying in the same plane:

$\vec{A} = \lambda\vec{a} + \mu\vec{b}$

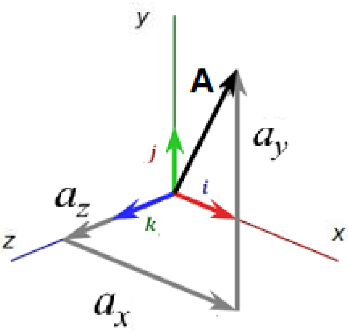
where λ and μ are real number

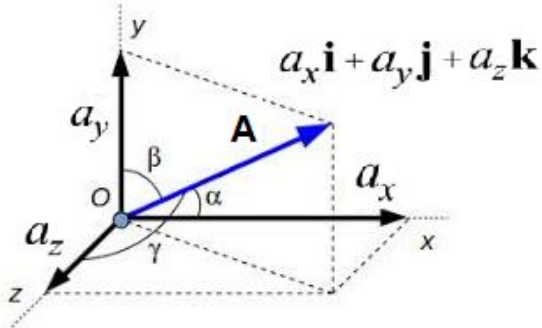
Any vector $\vec{A}$ can be expressed as a linear combination of the three unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$. where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are the unit vectors along X, Y and Z axes respectively.

Any vector $\vec{A}$ may have nonzero projections along X,Y, Z axes and we can resolve it into three parts i.e., along the X, Y and Z axes. If α, β, γ be the angles made by the vector a with the three axes respectively, we get

$$\boxed{\vec{A} = A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k} = a\cos\alpha\,\hat{\imath} + a\cos\beta\,\hat{\jmath} + a\cos\gamma\,\hat{k}}$$

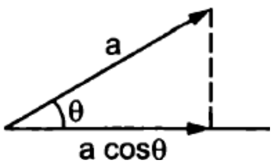
And $|\vec{A}| = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$ or $a = \sqrt{(a\cos\alpha)^2 + (a\cos\beta)^2 + (a\cos\gamma)^2}$





The magnitude (a cosα) is called the component of a along X-axis, (a cosβ) is called the component along Y-axis and (a cosγ) is called the component along Z-axis.

In general, the component of a vector $\vec{a}$ along a direction making an angle θ with it is a cos θ (fig) which is the projection of $\vec{a}$ along the given direction.



We can easily add or subtract two vectors if we know their components along the rectangular coordinate axes. Let us have

$\vec{A} = A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$, $\vec{B} = B_x\,\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k}$ and $\vec{R} = \vec{A} \pm \vec{B}$

Then

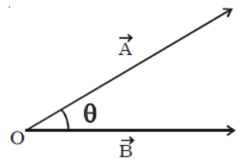
$$\boxed{\vec{R} = (A_x \pm B_x)\,\vec{\imath} + (A_y \pm B_y)\,\vec{\jmath} + (A_z \pm B_z)\,\vec{k}.}$$

- **Multiplication of two vectors**

Multiplication of a vector by another vector does not follow the laws of ordinary algebra. There are two types of vector multiplication

[i] Scalar product and [ii] Vector product.

[i] Scalar product or Dot product of two vectors:



Scalar product of two vectors

If the product of two vectors is a scalar, then it is called scalar product. If $\vec{A}$ and $\vec{B}$ are two vectors, then their scalar product is written as $\vec{A}\,.\,\vec{B}$ and read as $\vec{A}$ dot$\vec{B}$. Hence scalar product is also called dot product. This is also referred as inner or direct product.

The scalar product of two vectors is a scalar, which is equal to the product of magnitudes of the two vectors and the cosine of the smaller angle between them. The scalar product of two vectors $\vec{A}$ and $\vec{B}$ may be expressed as

$\vec{A}\,.\,\vec{B} = |\vec{A}|\;|\vec{B}|\cos\theta$ where $|\vec{A}|$ and $|\vec{B}|$ are the magnitudes of $\vec{A}$ and $\vec{B}$ respectively and θ is the angle between $\vec{A}$ and $\vec{B}$ as shown in Fig.

Properties of Scalar Product

(i) Scalar product is commutative, i.e., $\vec{A}\,.\,\vec{B} = \vec{B}\,.\,\vec{A}$

(ii) Scalar product is distributive, i.e., $\vec{A}\,.\,(\vec{B} + \vec{C}) = A\,.\,\vec{B} + \vec{A}\,.\,\vec{C}$

(iii) Scalar product of two perpendicular vectors is zero. $\vec{A}\,.\,\vec{B} = AB\cos 90° = O$

(iv) Scalar product of two parallel vectors is equal to the product of their magnitudes, i.e., $\vec{A}\,.\,\vec{B} = AB\cos 0° = AB$

(v) Scalar product of a vector with itself is equal to the square of its magnitude, i.e., $\vec{A}\,.\,\vec{A} = AA\cos 0° = A^2$

(vi) Scalar product of orthogonal unit vectors $\hat{\mathbf{i}}.\,\hat{\mathbf{i}} = \hat{\mathbf{j}}.\,\hat{\mathbf{j}} = \hat{\mathbf{k}}.\,\hat{\mathbf{k}}$ = 1x1xcos 0°= 1 and $\hat{\mathbf{i}}.\,\hat{\mathbf{j}} = \hat{\mathbf{j}}.\,\hat{\mathbf{k}} = \hat{\mathbf{k}}.\,\hat{\mathbf{i}}$ =1x1xcos 90°= 0

(vii) Scalar product in Cartesian coordinates $\vec{A}\,.\,\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\,.\,(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) = A_xB_x + A_yB_y + A_zB_z$

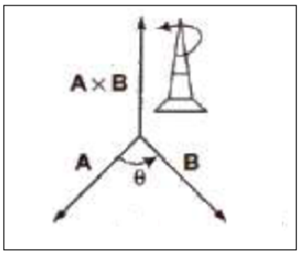
[ii] Vector product or Cross product of two vectors:

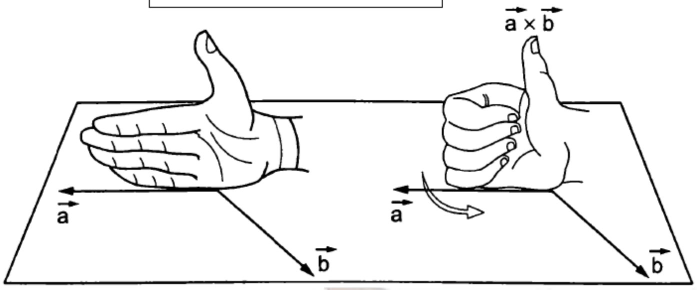
If the product of two vectors is a vector, then it is called vector product. If $\vec{A}$ and $\vec{B}$ are two vectors then their vector product is written as $\vec{A} \times \vec{B}$ and read as $\vec{A}$ cross $\vec{B}$. This is also referred as outer product.
The vector product or cross product of two vectors is a vector whose magnitude is equal to the product of their magnitudes and the sine of the smaller angle between them and the direction is perpendicular to a plane containing the two vectors.
If θ is the smaller angle through which $\vec{A}$ should be rotated to reach $\vec{B}$, then the cross product of $\vec{A}$ and $\vec{B}$ (Fig.) is expressed as,

$$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}|\sin\theta\,\hat{n} = \vec{C}$$

where $|\vec{A}|$ and $|\vec{B}|$ are the magnitudes of $\vec{A}$ and $\vec{B}$ respectively. $\vec{C}$ is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$.

Direction of Vector Cross Product:
When $\vec{C} = \vec{A} \times \vec{B}$, the direction of $\vec{C}$ is at right angles to the plane containing the vectors A and B. The direction is determined by the right hand screw rule and right hand thumb rule.





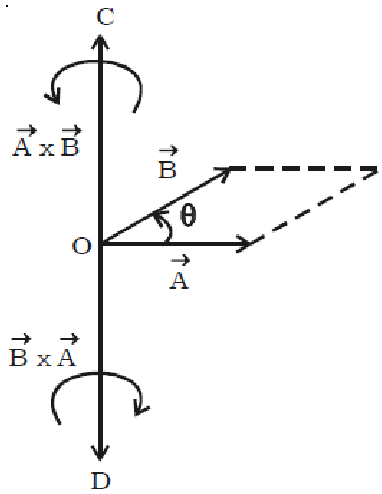
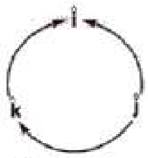


Fig Vector product of two vectors

(i) Right Hand Screw Rule : Rotate a right handed screw from first vector (A) towards second vector (B). The direction in which the right handed screw moves gives the direction of vector ($\vec{C}$).
(ii) Right Hand Thumb Rule : Curl the fingers of your right hand from $\vec{A}$ to $\vec{B}$. Then, the direction of the erect thumb will point in the direction of $\vec{A} \times \vec{B}$.
In Fig., The direction of $\vec{C}$ is along the direction in which the tip of a screw moves when it is rotated from $\vec{A}$ to $\vec{B}$. Hence $\vec{C}$ acts along OC. By the same argument, $\vec{B} \times \vec{A}$ acts along OD.

Properties of Vector Product:
(i) Vector product is not commutative, i.e., $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ $[(\vec{A} \times \vec{B}) = -(\vec{B} \times \vec{A})]$
(ii) Vector product is distributive, i.e., $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
(iii) Vector product of two parallel vectors is zero, i.e., $\vec{A} \times \vec{B} = AB \sin 0° = 0$
(iv) Vector product of any vector with itself is zero. $\vec{A} \times \vec{A} = AA \sin 0° = 0$
(v) Vector product of orthogonal unit vectors,
$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 1x1x\cos 90° = 0$
and $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$, $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$;
$\hat{\mathbf{j}} \times \mathbf{i} = -\hat{\mathbf{k}}$, $\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$, $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$.



(vi) Vector product in cartesian coordinates
$\vec{A} \times \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) = (A_yB_z - A_zB_y)\,\hat{\mathbf{i}} + (A_xB_z - A_zB_x)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\hat{\mathbf{k}}$

Co-ordinate Geometry

→ STRAIGHT LINES :
1) Distance between two points (x_1, y_1) &(x_2,y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2) Equation of a line having slope 'm' & Y- intercept 'c' is given by **y= mx + c**

3) Equation of a line passing through a given point (x_1, y_1) & having slope 'm' is given by $\mathbf{y - y_1 = m(x - x_1)}$

4) Equation of a line passing through two given points (x_1, y_1) & (x_2, y_2) is given by $y - y_1 = (x - x_1)\frac{(y_2-y_1)}{(x_2-x_1)}$

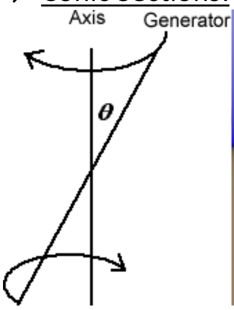
5) Equation of a line making intercepts 'a' & 'b' from the co-ordinate axis is given by $(x/a) + (y/b) = 1$

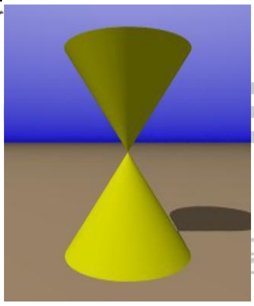
6) Equation of a line in normal form is given by $x \cos\alpha + y \sin\alpha = p$

7) Angle between two lines having slopes 'm_1' & 'm_2' is given by $\tan\theta = (m_1 - m_2)/(1 + m_1.m_2)$

8) Two lines will be parallel if $\mathbf{m_1 = m_2}$ & perpendicular if $\mathbf{m_1.m_2 = -1}$

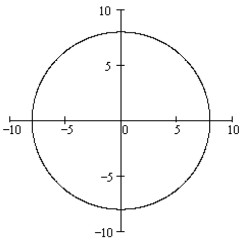
→ Conic Sections:

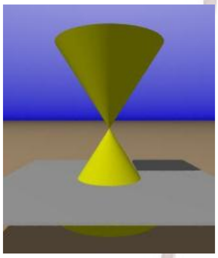




A conic shape is generated by intersecting two lines at a point and rotating one line around the other while keeping the angle between the lines constant.
The resulting collection of points is called a right circular cone. The two parts of the cone intersecting at the vertex are called nappes.

A conic section (or a conic) is a curve of intersection of a right circular cone of two nappes & a plane.
(i) The plane can intersect the cone at the vertex resulting in a point.
(ii) The plane can intersect the cone perpendicular to the axis resulting in a circle.



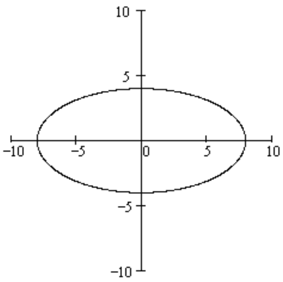


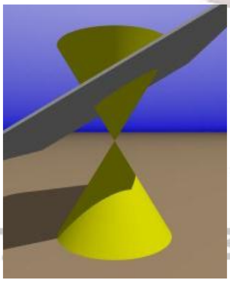
Standard equations of circle:
The equation of a circle having centre at (h ,k) & radius 'r' is given by

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$

If the centre is at the origin then the equation becomes $x^2 + y^2 = r^2$

(ii) The plane can intersect one nappe of the cone at an angle to the axis resulting in an ellipse.



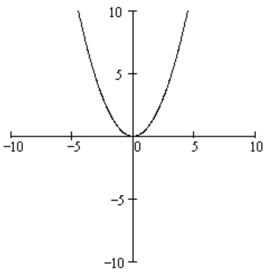


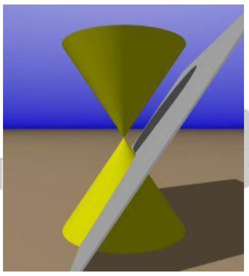
standard equations of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a^2 > b^2)$.............. horizontal ellipse

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ $(b^2 < a^2)$................ Vertical ellipse

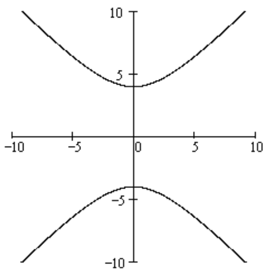
(iii) The plane can intersect one nappe of the cone at an angle to the axis resulting in a parabola.

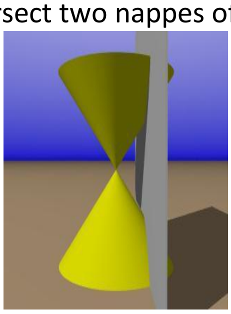




standard equations of parabola:
1. $y^2 = 4ax$ 2. $y^2 = -4ax$ 3. $x^2 = 4ay$ and 4. $x^2 = -4ay$

(iv) The plane can intersect two nappes of the cone resulting in a hyperbola.





standard equations of hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $(a^2 > b^2)$........ horizontal hyperbola

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ $(b^2 > a^2)$... .... Vertical hyperbola